Graph Theory

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There is a group of six people and not everyone is friends with everyone else.

**Friends**

Adam: Ben, Cindy  
Ben: Adam, Cindy  
Cindy: Adam, Ben, Dave, Edward, Frank  
Dave: Cindy  
Edward: Cindy, Frank  
Frank: Cindy, Edward

How would you draw something to depict these friendships?
Friendship Graphs

This is one way that these relationships can be diagrammed.

<table>
<thead>
<tr>
<th>Friends</th>
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<td>A: B, C</td>
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<tr>
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</tr>
<tr>
<td>C: A, B, D, E, F</td>
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<td>D: C</td>
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![Friendship Graph Diagram](image)
What is a Graph?

A graph $G$ is an ordered pair $(V(G), E(G))$, where $V(G)$ is a nonempty finite set and $E(G)$ is a set of 2-element subsets of $V(G)$.

Vertex and Edge Sets

$V(G)$ is called the vertex-set of $G$.
- The elements or “things” in $V(G)$ are the vertices of $G$.
- $|V(G)|$ (number of vertices in a graph $G$) is called the order of $G$.

$E(G)$ is called the edge-set of $G$.
- The elements of “things” in $E(G)$ are the edges of $G$.
- $|E(G)|$ (number of edges in a graph $G$) is called the size of $G$. 
An Example

An example of a Graph of Order 5 and Size 6:

\[ G = (\{0, 1, 2, 3, 4\}, \{\{0, 1\}, \{0, 2\}, \{0, 3\}, \{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}\}) \]
An Example

An example of a Graph of Order 5 and Size 6:

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This is a \textit{drawing} of \( G \):
More Terminology

Definitions

- A vertex is **incident** with an edge if the edge is connected to that edge and vice versa.
- A vertex is **adjacent** with another vertex if they are both incident with the same edge.
- An edge is **adjacent** with another edge if both edges are incident with the same vertex.

Coloring

We often talk about coloring a graph’s vertices. **Coloring** is assigning a color to a vertex, and we usually do this to group the vertices into subsets. To **properly color** a graph, no two adjacent vertices are assigned the same color.
Bipartite Graphs

Definition

A graph $G$ is bipartite if $V(G)$ can be colored properly so that no two connected edges are the same color with a minimum of two colors, say red and black. Therefore, every edge has one red end-vertex and one black end-vertex.
The degree of a vertex is the number of edges that are connected to that vertex, and is denoted by \( \text{deg}(v) \).

A graph \( G \) is regular if every vertex of \( G \) has the same degree.

- If \( \text{deg}(v) = n \) for every vertex \( v \) in \( V(G) \), then \( G \) is called \( n \)-regular.
- This example is 3-regular because the degree of each vertex is 3.
Complete Graphs

A graph is a complete graph if every vertex is adjacent to every other vertex.

In other words, there is an edge between every vertex.

A complete graph with \( n \) vertices (and \( n \) edges) is denoted \( K_n \).

Complete Bipartite Graphs

A complete bipartite graph with \( m \) and \( n \) vertices in each of the vertex subsets respectively is denoted \( K_{m,n} \) and has \( m \times n \) edges.

In \( K_{m,n} \), each vertex in the bottom set is adjacent to every vertex in the top set, but not adjacent to any vertices within the bottom vertex set.
Cycles

Definition

A cycle is a connected 2-regular graph. A cycle is determined by the number of vertices. For example, a cycle with 8 vertices is called an 8-Cycle or a $C_8$.

Observations

- All cycles of the same size are isomorphic (That is, structurally the same).
- All cycles with size a multiple of two are bipartite.

Example:
Trees

Definition

A **tree** is a graph that does not contain any cycles.

Types of Trees

- A **path** is a tree whose vertices except for two have degree two and those two exclusions have degree one.
- A **caterpillar** is a tree which if you chop off all of its legs is a path.
- A **lobster** is a tree which if you “chop off” all of it’s legs is a caterpillar.
- A **star** is a tree with one vertex which is adjacent with every other vertex

Theorem

All trees are bipartite. (Skiena, 1990)
Paths
Caterpillars
Lobsters
Stars
Graph Labelings

Labeling

- Let $G$ be a graph with $n$ edges. A labeling of $G$ is a one-to-one function from $V(G)$ to the set of nonnegative integers.

- In other words, a labeling of $G$ is assigning a number to each vertex. Labeling allows us to discuss edge length.

This is an example of a labeled graph $G$:

Here is $G$ placed inside $K_5$:
Length of an edge in $K_n$

**Definition**

Let $V(K_n) = \mathbb{Z}_n$ and place the vertices of $K_n$ around an $n$-gon.

- The **label** of an edge $\{i, j\}$ is $|i - j|$.

- The **length** of $\{i, j\}$ is the shortest distance from $i$ to $j$ “around” the polygon:
  
  $$\text{length}(\{i, j\}) = \min(\{|i - j|, n - |i - j|\}).$$

- Edge $\{i, j, \}$ is a **wrap-around edge** (denoted with *) if its length is not equal to its label.
Length of an edge in $K_n$

**Note**

The number of edges of length $i$ is dependent on $n$

- If $n = 2t + 1$, then $K_n$ consists of $n$ edges of length $i$ for each $i \in \{1, 2, \ldots, t\}$.

- In $n = 2t$, then $K_n$ consists of $n$ edges of length $i$ for each $i \in \{1, 2, \ldots, t - 1\}$ and $t$ edges of length $t$ (these form a 1-factor in $K_{2t}$).
Length of an edge in $K_{n,n}$

- Let $V(K_{n,n}) = (\mathbb{Z}_n \times \{0\}) \cup (\mathbb{Z}_n \times \{1\})$.
- Denote edge $\{(i,0), (j,1)\}$ in $K_{n,n}$ by $(i,j)$.
- The length of edge $(i,j)$ is $j - i$ if $j \geq i$ and $n + j - i$, otherwise.
- Note that $E(K_{n,n})$ consists of $n$ edges of length $i$ for $0 \leq i \leq n - 1$. 

\[\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 \\
\end{array}\]

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Graph Decomposition and Designs

**G-decomposition**

Let $G$ and $H$ be graphs (or multigraphs) with $G$ a subgraph of $H$. A **$G$-decomposition** of $H$ is a partition of the edge set of $H$ into subgraphs isomorphic to $G$ (called $G$-blocks).

**$(H, G)$-design**

An $G$-decomposition of $H$ is also called an $(H, G)$-design.

This is an example of a $G$-decomposition where $H$ is $K_4$ and $G$ is a $P_4$ ($G$ is shown below):
Cyclic Decompositions and Clicking

**Clicking**

Let $V(K_n) = \mathbb{Z}_n$, and let $G$ be a subgraph of $K_n$. By clicking $G$, we mean applying the isomorphism $i \rightarrow i + 1$ to $V(G)$.

**Cyclic Decompositions**

A $G$-decomposition of $K_n$ is cyclic if clicking $G$ preserves the $G$-blocks of the decomposition.

A cyclic $K_3$ decomposition of $K_7$ where $V(K_7) = \{0, 1, \ldots, 6\}$:

$\rightarrow (0, 1, 3)$
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$\rightarrow (0, 1, 3) \rightarrow (1, 2, 4) \rightarrow (2, 3, 5)$
$\rightarrow (3, 4, 6) \rightarrow (4, 5, 0)$
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\]
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$\rightarrow (0, 1, 3) \rightarrow (1, 2, 4) \rightarrow (2, 3, 5) \rightarrow (3, 4, 6) \rightarrow (4, 5, 0) \rightarrow (5, 6, 1) \rightarrow (6, 0, 2)$
Cyclic Decompositions in Complete Bipartite Graphs

Clicking

Let

\[ V(K_{n,n}) = (\mathbb{Z}_n \times \{0\}) \cup (\mathbb{Z}_n \times \{1\}) \].

By clicking \( G \), we mean applying the isomorphism \( (i, j) \to (i + 1, j) \) to \( V(G) \).

Cyclic Decompositions

A \( G \)-decomposition of \( K_{n,n} \) is cyclic if clicking \( G \) preserves the \( G \)-blocks of the decomposition.

A cyclic \( S_5 \) decomposition of \( K_{5,5} \).
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Rosa’s Original Labelings
for a graph $G$ with $n$ edges

$\rho$-labeling

Vertex labels from $\{0, 1, \ldots, 2n\}$; one edge of each length.
There exists a cyclic $(K_{2n+1}, G)$-design if and only if $G$ has a $\rho$-labeling.
Rosa’s Original Labelings for a graph $G$ with $n$ edges

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### Rosa’s Original Labelings

for a graph $G$ with $n$ edges

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<td><strong>$\alpha$-labeling</strong></td>
<td>$G$ must be bipartite. All vertex labels from one part in the vertex partition must be less than those of the other part.</td>
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\(\alpha\)-labelings

**Theorems**

- Every cycle with a size that is a multiple of 4 has an \(\alpha\)-labeling.
- Every path, caterpillar, and lobster has an \(\alpha\)-labeling.

**Conjecture**

The union of two cycles with size that is 2 more than a multiple of 4 has an \(\alpha\)-labeling.

\(\alpha\)-labelings, although very restrictive are very useful. One of their better applications is they allow for something called stretching. We denote the largest number in the top partition of an \(\alpha\)-labeling with the Greek letter \(\lambda\).
Examples of Alpha Labelings
Examples of Alpha Labelings

\[ \lambda = 0 \]

0
3 2 1
12 11 10 9 8 7

\[ \lambda = 6 \]

0 1 4 2 5 6
12 11 10 9 8 7

\[ \lambda = 1 \]

0 1
4 3 2

\[ \lambda = 3 \]

0 1 2 3
8 7 5 4

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Examples of Alpha Labelings

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Examples of Alpha Labelings

\begin{align*}
\lambda &= 0 \\
&\quad \begin{array}{cccccc}
0 & 1 & 4 & 2 & 5 & 6 \\
3 & 2 & 1 & 0 & 1 & 4 & 2 & 5 & 6 \\
\end{array} \\
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0 & 1 \\
4 & 3 & 2 \\
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\end{array}
\end{align*}
Examples of Alpha Labelings

\[ \lambda = 0 \]

\[ \lambda = 6 \]

\[ \lambda = 1 \]

\[ \lambda = 3 \]
You can use an $\alpha$-labeling as a template in order to not just decompose a $K_{2n+1}$, but also $K_{2nx+1}$ where $x$ is any natural number.

**Definition**

In other words, stretching an $\alpha$-labeling allows for a G-Block to decompose an infinite number of graphs. We call this process stretching, because you simply "stretch the edge lengths by the size of the original graph.

**Example:**

![Graph example](image-url)
You can use an $\alpha$-labeling as a template in order to not just decompose a $K_{2n+1}$, but also $K_{2nx+1}$ where $x$ is any natural number.

**Definition**

In other words, **stretching** an $\alpha$-labeling allows for a G-Block to decompose an infinite number of graphs. We call this process "stretching, because you simply "stretch the edge lengths by the size of the original graph."

**Example:**

![Graphs](image-url)
We know that there is an $\alpha$-labeling of the union of any two cycles with size $4r$ where $r$ is a natural number.

“Gee, I wonder...”

- Is there an $\alpha$-labeling of any two cycles with size $4s + 2$ where $s$ is a natural number?
- Is there an $\alpha$-labeling of the union of any number of cycles with size $4r$ where $r$ is a natural number?
- Can you take the union of multiple classifications of graphs with $\alpha$-labelings to generate a labeling that will decompose a complete bipartite graph?
Answer

YES!
Theorem

Let $G_i$ be a bipartite graph with size $m_i$, $\alpha$-labeling $f_i$, critical value $\lambda_i$, and vertex bipartition $\{A_i, B_i\}$ for all $i$ such that $1 \leq i \leq n$. Also, let $G = G_1 \cup G_2 \cup \cdots \cup G_n$. There exists a labeling of $G$ such that $G$ cyclically decomposes $K_{m+1,m+1} - F$ where $F$ is a 1-factor of $K_{m+1,m+1}$. 
Arrangement of $G_i$

If $n$ is even

Let $n = 2t$ for some positive integer, $t$. Without loss of generality we can assume

\[ \lambda_1 \geq \lambda_{t+1} \geq \lambda_2 \geq \lambda_{t+2} \geq \cdots \geq \lambda_t \geq \lambda_{2t}. \]
Arrangement of $G_i$

If $n$ is even

Let $n = 2t$ for some positive integer, $t$. Without loss of generality we can assume

$$\lambda_1 \geq \lambda_{t+1} \geq \lambda_2 \geq \lambda_{t+2} \geq \cdots \geq \lambda_t \geq \lambda_{2t}.$$ 

If $n$ is odd

Let $n = 2t - 1$ for some positive integer, $t$. Without loss of generality we can assume

$$\lambda_1 \geq \lambda_{t+1} \geq \lambda_2 \geq \lambda_{t+2} \geq \cdots \geq \lambda_{t-1} \geq \lambda_{2t-1} \geq \lambda_t.$$
Example when $n$ is even

For $G_i$ such that $1 \leq i \leq t$

$$f'_i(v) = \begin{cases} 
    f_i(v) + \sum_{j=1}^{i-1} (\lambda_j + 1), & v \in B_i \\
    f_i(v) + \sum_{j=1}^{i-1} (\lambda_j + 1) + m - \sum_{j=1}^{i-1} (m_j + m_{2t-j+1}) - m_i, & v \in A_i 
\end{cases}$$
Example when $n$ is even

For $G_i$ such that $t + 1 \leq i \leq 2t$

$$f'_i(v) = \begin{cases} 
    f_i(v) + \sum_{j=t+1}^{i-1} (\lambda_j + 1), & v \in A_i \\
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\end{cases}$$
Example when $n$ is even
Example when $n$ is odd

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Example when $n$ is odd

For $G_i$ such that $t + 1 \leq i \leq 2t - 1$

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\end{cases}$$
Example when $n$ is odd
Corollaries

**Corollary 1**

Since we are able to do this with any number of graphs that admit an $\alpha$-labeling, then we can produce a labeling that decomposes a complete bipartite graph using the union of any two cycles whose sizes are congruent to 0 (mod 4).

**Corollary 2**

Our labeling, although it is based on $\alpha$-labelings, is not an $\alpha$-labeling itself. However, it does allow for stretching. Therefore, we can use a G-Block to decompose any $K_{mx+1, mx+1}$ complete bipartite graph; where $m$ is the size of the G-Block, and $x$ is any natural number.
THANK YOU!